equations in a Newtonian central-force field for a circular nominal trajectory. Following Eq. (5) he states that "Eqs. (5) are applicable to perturbations of a general elliptical trajectory. However, the integration to obtain δr and $\delta\theta$ becomes a difficult task because of dependence on time of the unperturbed parameters r and θ." It will be shown that the solution by elementary functions is possible if the eccentric anomaly E is used as an independent variable. The coordinate system is chosen in such a way that the z axis is perpendicular to the plane of the unperturbed motion. Then, the rigorous differential equation of motion in spatial cylindrical coordinates (r, θ, z) are

$$\ddot{r} + \frac{Kr}{(r^2 + z^2)^{3/2}} - r\dot{\theta}^2 = 0 \qquad \frac{d}{dt}(r^2\,\dot{\theta}) = 0 \tag{1}$$

and

$$\ddot{z} + \frac{Kz}{(r^2 + z^2)^{3/2}} = 0 \tag{2}$$

The solution corresponding to the nominal initial conditions is called $r = r_0$, $\theta = \theta_0$. $z_0 \equiv 0$ due to the choice of the coordinate system. The solution corresponding to the perturbed initial condition is denoted by

$$r = r_0 + \delta r \qquad \theta = \theta_0 + \delta\theta \qquad z = \delta z \tag{3}$$

In a straightforward manner, one now could derive equations linear in δr, $\delta\theta$, and δz. But it is immediately clear from the character of Eq. (1) that such a system of differential equations of δr and $\delta\theta$ would be coupled. To avoid this difficulty, the energy integral[2] is used:

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 - \frac{2K}{(r^2 + z^2)^{1/2}} + \frac{K}{a_0} = C_3 \tag{4}$$

where a_0 is the major axis of the unperturbed elliptical motion, K the gravitational constant of the central force field, and C_3 a constant of integration which vanishes if $r = r_0$ and $\theta = \theta_0$. Multiplying the differential equations for r in (1) by r, adding (4), and linearizing yields

$$[d^2(r_0\delta r)/dt^2] + (K/r_0^3)(r_0\,\delta r) = C_3 \tag{5}$$

On the other hand, one obtains by a twofold subtraction of Eq. (5) from the linearized energy integral

$$\frac{d\delta\theta}{dt} = \frac{1}{na_0^2(1 - e^2)^{1/2}}\left[\frac{d}{dt}\left(2\,\frac{d(r_0\delta r)}{dt} - \delta r\,\frac{dr_0}{dt}\right) - \frac{3C_3}{2}\right] \tag{6}$$

Finally, direct linearization of Eq. (2) yields

$$[d^2(\delta z)/dt^2] + (K/r_0^3)\delta z = 0 \tag{7}$$

The integration problem is reduced to the solution of one differential equation of second order, namely

$$(d^2q/dt^2) + (K/r_0^3)q = C \tag{8}$$

With the well-known relation

$$\begin{aligned} ndt &= [1 - e\cos(E' + E_0)]dE' \\ n^2 &= K/a_0^3 \qquad E' = E - E_0 \end{aligned} \tag{9}$$

where e is the eccentricity, E the eccentric anomaly, and E_0 the eccentric anomaly corresponding to the beginning of the motion, one derives from the homogeneous part of Eq. (8)

$$[1 - e\cos(E' + E_0)](d^2q/dE'^2) - e\sin(E' + E_0)\,(dq/dE') + q = 0 \tag{10}$$

A fundamental solution of (10) with the Wronskian equal to one is

$$\begin{aligned} q_1 &= n^{-1/2}\,[\cos E' - e\cos E_0] \\ q_2 &= n^{-1/2}\,[\sin E' + e\sin E_0] \end{aligned} \tag{11}$$

Thus, the general solution of (8) can be written as

$$q = C_1q_1 + C_2q_2 + C\left\{q_2\int_0^{E'} q_1\,dt - q_1\int_0^{E'} q_2\,dt\right\}$$
$$E' = E - E_0 \tag{12}$$

From Eqs. (9) and (11) it follows that the integrals in (12) can be expressed as trigonometric functions of E and rational functions of e. For the sake of brevity, the explicit expressions are not written down, but it might be pointed out that the six constants of integration can be determined independently from each other if the quantity $E' = E - E_0$ is introduced. This follows easily from the character of Eqs. (6, 7, and 12). For an application of this classical method in perturbed central force fields, see Ref. 3.

References

[1] Wisneski, M. L., "Error matrix for a flight on a circular orbit," ARS J. **32**, 1416–1418 (1962).

[2] Brouwer, D. and Clemenc, G. M., *Methods of Celestial Mechanics* (Academic Press, New York and London, 1961).

[3] Dusek, H. M., "Theory of error propagation in astro-inertial guidance systems for low-thrust earth orbital missions," ARS Preprint 2683-62 (November 1962).

Reply to Comment by H. M. Dusek

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THE comment of Mr. Dusek and his ARS meeting paper are very interesting. It appears that a manageable solution of Eqs. (5) of his Ref. 1 can be obtained if the technique of his Ref. 2 for dealing with such equations is employed. The technique uses eccentric anomaly as an independent variable. However, to express perturbations in terms of the true anomaly rather than the eccentric anomaly, the conversion involves a series expansion. Thus an additional approximation, strictly valid for low-eccentricity orbits, has to be made when only the first few terms are retained.

In Dusek's Ref. 1 signs in two places are incorrect. Referring to the last two "error matrices," the signs in front of expressions in positions 2 and 3 should be plus.

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Response to Author's Reply

Hermann M. Dusek*
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THE introduction of the true anomaly is pertinent neither to my original comment nor to Wisneski's original paper. Wisneski states, "However, to express perturbations in terms of the true anomaly rather than the eccentric anomaly, the conversion involves a series expansion." This statement, in itself, certainly is true if one follows the classical astronomical practice (see Ref. 2, pp. 62–65, of the original comment). However, to this author's knowledge, series expansions for arbitrary eccentricities of the unperturbed orbit in connection with this particular problem can be avoided.

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The reader is referred to the following source: Shapiro, I. I., *The Prediction of Ballistic Missile Trajectories from Radar Observations* (McGraw-Hill Book Co. Inc., New York, 1958), pp. 93–98.

Comment on "Orbit Decay Characteristics Due to Drag"

JAIN-MING WU*
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A RECENT paper by Parsons[1] calculates the rotation of line of apsides and the other orbital characteristics purely due to the atmospheric drag for both small and large values of the eccentricity ϵ of the trajectory. However, in his initial setup of the equations of motion, he neglected a dr/dt term which is of the order of ϵ. This can be shown from the Keplerian trajectory, the radius, r:

$$r = \frac{a(1-\epsilon^2)}{1-\epsilon\cos\theta} = \frac{[r^2(d\theta/dt)]^2}{g_0R^2(1-\epsilon\cos\theta)} \qquad (1)$$

where θ is the angle of polar coordinates, g_0 is the gravitational acceleration at sea level, R is the radius of the earth, and a is one half of the sum of perigee and apogee radius. So,

$$\frac{dr}{dt} = \frac{a(1-\epsilon^2)\epsilon\sin\theta}{(1-\epsilon\cos\theta)^2}\frac{d\theta}{dt} = \epsilon\sin\theta\left[\frac{g_0R^2}{a(1-\epsilon^2)}\right]^{1/2} \qquad (2)$$

Hence, dr/dt is the order of ϵ. Thus, his resulting solutions are in error for terms of order ϵ^2 and higher, and the final conclusions are valid only for orbits of small eccentricity.

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[1] Parsons, W. D., "Orbit decay characteristics due to drag," ARS J. **32**, 1876–1881 (1962).

Author's Reply to Comment by Jain-Ming Wu

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WU'S comment is well taken, as far as it goes. However, as pointed out in the subject paper,[1] the neglected term, $(dr/dt)^2$ [Eqs. (3) and (4)], is multiplied by the air density ρ, which decreases swiftly in an exponential manner as the $(dr/dt)^2$ values increase significantly. Moreover, the slight effect of neglecting the $(dr/dt)^2$ is to reduce the drag impulse so that the magnitude of the derived results may be slightly low.

At the risk of belaboring the point, an extreme numerical example, given below, shows the uselessness of retaining the $(dr/dt)^2$ terms. Wu shows that

$$\left(\frac{dr}{dt}\right)^2 = \frac{\epsilon^2 g_0 R^2}{a(1-\epsilon^2)}\sin^2\theta \qquad (1)$$

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[1] Parsons, W. D., "Orbit decay characteristics due to drag," ARS J. **32**, 1876–1881 (1962).

In a similar manner

$$\left(r\frac{d\theta}{dt}\right)^2 = g_0R^2\frac{(1+\epsilon\cos\theta)^2}{a(1-\epsilon^2)} \qquad (2)$$

The value of θ, where the total speed is in error by, say, 1% due to the neglect of the $(dr/dt)^2$, is calculated using Eqs. (1) and (2). Recognizing that the last term under the radical is very small compared to unity, expansion gives

$$\left[1 + \frac{(dr/dt)^2}{(rd\theta/dt)^2}\right]^{1/2} = 1.01 = 1 + \frac{1}{2}\frac{\epsilon^2\sin^2\theta}{(1+\epsilon\cos\theta)^2} \qquad (3)$$

Since Wu is concerned about the large eccentricities, Eq. (3) is studied, using the extreme case of the escape parabola where $\epsilon = 1$.

$$\frac{\sin^2\theta}{(1+\cos\theta)^2} = \frac{1}{50} = \frac{1-\cos\theta}{1+\cos\theta} \qquad (4)$$

This relation shows that the $(dr/dt)^2$ term becomes as important as 1% of the total velocity at $\theta \cong 16°$.

Consider next the altitude change that occurs during that 16° of motion. The equation of the parabola for the extreme case of perigee at the earth's surface is

$$r = 2R_0/(1+\cos\theta) \qquad (5)$$

The altitude is determined, using Eqs. (4) and (5) as

$$r - R_0 = R_0\left[\frac{1-\cos\theta}{1+\cos\theta}\right] = \frac{R_0}{50} \cong 69 \text{ naut miles} \qquad (6)$$

Since the density decreases by the factor e about every 23 naut miles, ρ is down by the factor $e^{-3} = \frac{1}{20}$.

Therefore, it seems reasonable to suggest that the error in describing the drag, due to neglecting the $(dr/dt)^2$ term, occurs at altitudes high enough to cause a negligible effect on the total drag pulse.

Comments on a Hanging Soap Film

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IN a recent note,[1] an apparent paradox has been encountered in the study of a soap film hanging on a horizontal circular frame. If one assumes that the "tensile force per unit length" within the film is constant at every point, the equations for static equilibrium of an element of area are not consistent. It is known, however (see, e.g., Ref. 2), that films, foams, etc., are stable in a gravitational field only if the surface energy (i.e., surface tension) is variable over the surface. The surface energy for a pure substance depends, essentially, on the temperature only, whereas for a liquid mixture it depends strongly on the relative concentrations of the constituents as well.

Films such as that under consideration are observed to be stable only if a liquid mixture, e.g., a soap solution, is used. Hence, when the equilibrium conditions are investigated, variations in the surface tension must be accounted for; the simplest example is a flat vertical film. In the present case, one may take T (in the notation of Ref. 1) to be a function of r only. The equations of horizontal and vertical equilibrium subsequently are found to be

$$\frac{d\phi}{dr} + \frac{\tan\phi}{r}\left(1 - \frac{d\ln T}{d\ln r}\right) = 0$$

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